

LOW DIMENSIONAL LINEAR REPRESENTATIONS OF THE MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

BLAŻEJ SZEPIETOWSKI

ABSTRACT. Suppose that f is a homomorphism from the mapping class group $\mathcal{M}(N_{g,n})$ of a nonorientable surface of genus g with n boundary components, to $\mathrm{GL}(m, \mathbb{C})$. We prove that if $g \geq 5$, $n \leq 1$ and $m \leq g - 2$, then f factors through the abelianization of $\mathcal{M}(N_{g,n})$, which is $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $g \in \{5, 6\}$ and \mathbb{Z}_2 for $g \geq 7$. If $g \geq 7$, $n = 0$ and $m = g - 1$, then either f has finite image (of order at most two if $g \neq 8$), or it is conjugate to one of four “homological representations”. As an application we prove that for $g \geq 5$ and $h < g$, every homomorphism $\mathcal{M}(N_{g,0}) \rightarrow \mathcal{M}(N_{h,0})$ factors through the abelianization of $\mathcal{M}(N_{g,0})$.

1. INTRODUCTION

For a compact surface F , its *mapping class group* $\mathcal{M}(F)$ is the group of isotopy classes of all, orientation preserving if F is orientable, homeomorphisms $F \rightarrow F$ equal to the identity on the boundary of F . A compact surface of genus g with n boundary components will be denoted by $S_{g,n}$ if it is orientable, or by $N_{g,n}$ if it is nonorientable. If $n = 0$ then we drop it in the notation and write simply S_g or N_g . The first integral homology group of F will be denoted by $H_1(F)$.

After fixing a basis of $H_1(S_g)$, the action of $\mathcal{M}(S_g)$ on $H_1(S_g)$ gives rise to a homomorphism $\mathcal{M}(S_g) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$, which is well known to be surjective, and whose kernel is known as the Torelli group. Gluing a disc along each boundary component of $S_{g,n}$ induces an epimorphism $\mathcal{M}(S_{g,n}) \rightarrow \mathcal{M}(S_g)$, and by composing it with $\mathcal{M}(S_g) \rightarrow \mathrm{Sp}(2g, \mathbb{Z})$, and then with the inclusion $\mathrm{Sp}(2g, \mathbb{Z}) \hookrightarrow \mathrm{GL}(2g, \mathbb{C})$ we obtain the map $\Phi: \mathcal{M}(S_{g,n}) \rightarrow \mathrm{GL}(2g, \mathbb{C})$. Recently, the following two results were proved by J. Franks, M. Handel and M. Korkmaz.

Theorem 1.1 ([6, 14]). *Let $g \geq 2$, $m \leq 2g - 1$ and let $f: \mathcal{M}(S_{g,n}) \rightarrow \mathrm{GL}(m, \mathbb{C})$ be a homomorphism. Then f is trivial if $g \geq 3$, and $\mathrm{Im}(f)$ is a quotient of \mathbb{Z}_{10} if $g = 2$.*

Supported by NCN grant nr 2012/05/B/ST1/02171.

We say that two homomorphism f_1, f_2 from a group G to a group H are *conjugate* if there exists $h \in H$ such that $f_2(x) = hf_1(x)h^{-1}$ for $x \in G$.

Theorem 1.2 ([15]). *For $g \geq 3$, every nontrivial homomorphism $f: \mathcal{M}(S_{g,n}) \rightarrow \mathrm{GL}(2g, \mathbb{C})$ is conjugate to the map Φ .*

In this paper we prove analogous results for $\mathcal{M}(N_g)$. Fix $g \geq 3$. Let R_g denote the quotient of $H_1(N_g)$ by its torsion. Hence, R_g is a free \mathbb{Z} -module of rank $g - 1$. There is covering $P: S_{g-1} \rightarrow N_g$ of degree two. By a theorem of Birman and Chillingworth [3], $\mathcal{M}(N_g)$ is isomorphic to the subgroup of $\mathcal{M}(S_{g-1})$ consisting of the isotopy classes of orientation preserving lifts of homeomorphisms of N_g , which gives an action of N_g on $H_1(S_{g-1})$. Let $K_g \subset H_1(S_{g-1})$ be the kernel of the composition of the induced map $P_*: H_1(S_{g-1}) \rightarrow H_1(N_g)$ with the canonical projection $H_1(N_g) \rightarrow R_g$. Then K_g is $\mathcal{M}(N_g)$ -invariant subgroup of rank $g - 1$ and we have two homomorphisms

$$\Psi_1: \mathcal{M}(N_g) \rightarrow \mathrm{GL}(K_g) \quad \text{and} \quad \Psi_2: \mathcal{M}(N_g) \rightarrow \mathrm{GL}(H_1(S_{g-1})/K_g),$$

which after fixing bases will be treated as representations of $\mathcal{M}(N_g)$ in $\mathrm{GL}(g - 1, \mathbb{C})$. We will see that these representations are not conjugate, although $\ker \Psi_1 = \ker \Psi_2$.

Our first result is the following.

Theorem 1.3. *Suppose that $n \leq 1$, $g \geq 5$, $m \leq g - 2$ and $f: \mathcal{M}(N_{g,n}) \rightarrow \mathrm{GL}(m, \mathbb{C})$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is either \mathbb{Z}_2 or $\mathbb{Z}_2 \times \mathbb{Z}_2$, the latter case being possible only for $g = 5$ or 6 .*

Theorem 1.3 was proved in [14], in a more general setting of punctured surfaces, under additional assumption that $m \leq g - 3$ if g is even. Therefore, the only novelty of our result is that it also covers the case $m = g - 2$ for even g . As an application of Theorem 1.3 we prove the following result, which solves Problem 3.3 in [13].

Theorem 1.4. *Suppose that $g \geq 5$, $h < g$ and $f: \mathcal{M}(N_g) \rightarrow \mathcal{M}(N_h)$ is a nontrivial homomorphism. Then $\mathrm{Im}(f)$ is as in Theorem 1.3.*

Analogous theorem for mapping class groups of orientable surfaces was proved in [10], see also [1]. We will prove that both Theorem 1.3 and Theorem 1.4 fail for $g = 4$, by showing that there is a homomorphism from $\mathcal{M}(N_4)$ to $\mathcal{M}(N_3) \cong \mathrm{GL}(2, \mathbb{Z})$, whose image is isomorphic to the infinite dihedral group.

Suppose that $g \geq 7$. Then the abelianization of $\mathcal{M}(N_g)$ is \mathbb{Z}_2 and we denote by $\mathrm{ab}: \mathcal{M}(N_g) \rightarrow \mathbb{Z}_2$ the canonical projection. For $i = 1, 2$ we set $\Psi'_i = (-1)^{\mathrm{ab}} \Psi_i$. Our next result is the following.

Theorem 1.5. *Suppose that $g \geq 7$, $g \neq 8$ and $f: \mathcal{M}(N_g) \rightarrow \mathrm{GL}(g-1, \mathbb{C})$ is a nontrivial homomorphism. Then either $\mathrm{Im}(f) \cong \mathbb{Z}_2$, or f is conjugate to one of $\Psi_1, \Psi'_1, \Psi_2, \Psi'_2$.*

For $g = 8$ other representations of $\mathcal{M}(N_8)$ in $\mathrm{GL}(7, \mathbb{C})$ occur, related to the fact that there is an epimorphism $\epsilon: \mathcal{M}(N_8) \rightarrow \mathrm{Sp}(6, \mathbb{Z}_2)$ and the last group admits irreducible representations in $\mathrm{GL}(7, \mathbb{C})$ (see [24]). We prove the following result.

Theorem 1.6. *Suppose that $f: \mathcal{M}(N_8) \rightarrow \mathrm{GL}(7, \mathbb{C})$ is a homomorphism. Then one of the following holds.*

- (1) $\mathrm{Im}(f) \cong \mathbb{Z}_2$.
- (2) f or $(-1)^{\mathrm{ab}} f$ factors through $\epsilon: \mathcal{M}(N_8) \rightarrow \mathrm{Sp}(6, \mathbb{Z}_2)$.
- (3) f is conjugate to one of $\Psi_1, \Psi'_1, \Psi_2, \Psi'_2$.

To prove our theorems we use the ideas and results from [6, 14, 15] with necessary modifications. While the case of odd genus is relatively easy, the case of even genus requires much more effort. This phenomenon is typical for the mapping group of a nonorientable surface.

Throughout this paper we will often have to solve an equation of the form $L = R$, where L and R are products of matrices from $\mathrm{GL}(m, \mathbb{C})$ with some unknown coefficients. Although the dimension m is variable, the calculations of L and R always reduce to multiplication of blocks of size at most 7×7 . With some patience, such calculations could be done by hand, but it is definitely easier to use a computer. We used GAP, but of course, any program that performs symbolic operations on matrices, could be used as well.

2. NOTATION AND ALGEBRAIC PRELIMINARIES

Suppose that $m \geq 2$ is fixed. We denote by I_m the identity matrix of dimension m . We will sometimes write simply I , if m is clear from the context. We denote by E_{ij} the elementary matrix with 1 on the position (i, j) and 0 elsewhere. Suppose that M_1, \dots, M_k are nonsingular square matrices of dimensions m_1, \dots, m_k , where $m_1 + \dots + m_k = m$. Then we denote by $\mathrm{diag}(M_1, \dots, M_k)$ the $m \times m$ matrix with M_1, \dots, M_k on the main diagonal and zeros elsewhere. Set

$$V = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \widehat{V} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad W = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

For $2 \leq 2i \leq m$ we define

$$A_i = \mathrm{diag}(I_{2i-2}, V, I_{m-2i}), \quad B_i = \mathrm{diag}(I_{2i-2}, \widehat{V}, I_{m-2i}),$$

and for $2 \leq 2j \leq m - 2$,

$$C_j = \text{diag} (I_{2j-2}, W, I_{m-2-2j}).$$

The proof of the following lemma is straightforward and we leave it as an exercise (c.f. [15, Lemma 2.2]).

Lemma 2.1. *Suppose that $1 \leq k \leq l \leq m/2$ and $M \in \text{GL}(m, \mathbb{C})$ satisfies $A_i M = M A_i$, $B_i M = M B_i$ and $C_j M = M C_j$ for all i, j such that $k \leq i \leq l$, $k \leq j \leq l - 1$. Then M has the form*

$$\begin{pmatrix} * & 0 & * \\ 0 & \lambda I_{2(l-k+1)} & 0 \\ * & 0 & * \end{pmatrix},$$

for some $\lambda \in \mathbb{C}^*$, where the top-left λ of the block $\lambda I_{2(l-k+1)}$ is at the position $(2k - 1, 2k - 1)$.

Suppose that $L \in \text{GL}(m, \mathbb{C})$ and λ is an eigenvalue of L . Then we denote by $\#\lambda$ the multiplicity of λ . For $k \geq 1$ we denote by $E^k(L, \lambda)$ the space $\ker(E - \lambda I)^k$. Thus $E^1(L, \lambda)$ is the eigenspace of L with respect to λ , and it will be also denoted by $E(L, \lambda)$. Note that if $L' \in \text{GL}(m, \mathbb{C})$ commutes with L , then the spaces $E^k(L, \lambda)$ are L' -invariant for $k \geq 1$.

For $k \geq 2$ we denote by \mathfrak{S}_k the full symmetric group of the set $\{1, \dots, k\}$. It is generated by the transpositions $\sigma_i = (i, i + 1)$ for $1 \leq i \leq k - 1$. We will need the following result from the representation theory of the symmetric group, see for example [7, Exercise 4.14].

Lemma 2.2. *For $k \geq 5$, \mathfrak{S}_k has no irreducible representation (over \mathbb{C}) of dimension $1 < m < k - 1$. If $k \geq 7$, then \mathfrak{S}_k has two irreducible representations of dimension $k - 1$: the standard one and the tensor product of the standard and sign representations.*

3. MAPPING CLASS GROUP OF A NONORIENTABLE SURFACE

Let $n \in \{0, 1\}$ and $g \geq 2$. Let us represent $N_{g,n}$ as a sphere (if $n = 0$) or a disc (if $n = 1$) with g crosscaps. This means that interiors of g small pairwise disjoint discs should be removed from the sphere/disc, and then antipodal points in each of the resulting boundary components should be identified. Let us arrange the crosscaps as shown on Figure 1 and number them from 1 to g . For each nonempty subset $I \subseteq \{1, \dots, g\}$ let ξ_I be the simple closed curve shown on Figure 1. Note that ξ_I is two-sided if and only if I has even number of elements. In such case t_{ξ_I} will be the Dehn twist about γ_I in the direction indicated by arrows on Figure 1. We will write ξ_i instead of $\xi_{\{i\}}$. The following curves will play a special role and so we give them different names.

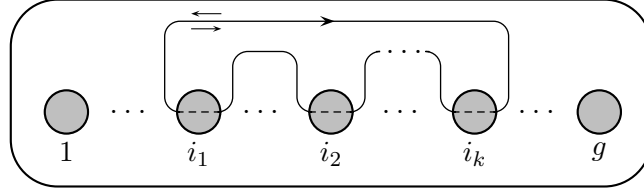


FIGURE 1. The surface $N_{g,n}$ and the curve ξ_I for $I = \{i_1, i_2, \dots, i_k\}$.

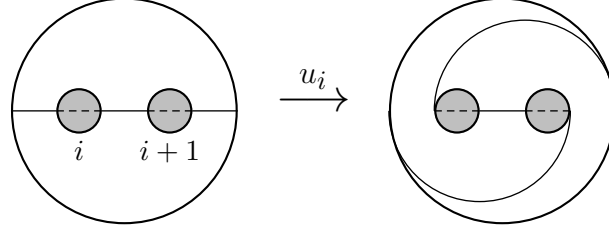


FIGURE 2. The crosscap transposition u_i .

- $\delta_i = \xi_{\{i, i+1\}}$ for $1 \leq i \leq g-1$,
- $\varepsilon_j = \xi_{\{1, 2, \dots, 2j\}}$ for $2 \leq 2j \leq g$.

Note that $\varepsilon_1 = \delta_1$.

For $1 \leq i \leq g-1$ we define the *crosscap transposition* u_i to be the isotopy class of the homeomorphism interchanging the i 'th and the $(i+1)$ 'st crosscaps as shown on Figure 2, and equal to the identity outside a disc containing these crosscaps.

The groups $\mathcal{M}(N_{1,n})$ are trivial for $n \leq 1$ by [5, Theorem 3.4], we have $\mathcal{M}(N_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by [16], and it follows from [3] that $\mathcal{M}(N_3) \cong \text{GL}(2, \mathbb{Z})$. For $g \geq 3$, a finite generating set for $\mathcal{M}(N_{g,n})$ was given in [4] for $n = 0$ and [19] for $n > 0$. For $n \leq 1$ this set can be reduced to the one given in the following theorem, which can be deduced from the main result of [18].

Theorem 3.1. *For $g \geq 4$ and $n \in \{0, 1\}$, $\mathcal{M}(N_{g,n})$ is generated by u_{g-1} , t_{ε_2} and t_{δ_i} for $1 \leq i \leq g-1$.*

If $n > 1$, then we consider $N_{g,n}$ as the result of gluing $S_{0,n+1}$ to $N_{g,1}$ along the boundary component. We will need the following relations, satisfied in $\mathcal{M}(N_{g,n})$. Those between Dehn twists are the well known disjointness and braid relations.

- (R1) $t_{\delta_i} t_{\delta_j} = t_{\delta_j} t_{\delta_i}$ for $|i - j| > 1$,
- (R2) $t_{\varepsilon_i} t_{\varepsilon_j} = t_{\varepsilon_j} t_{\varepsilon_i}$ for all i, j ,
- (R3) $t_{\varepsilon_i} t_{\delta_j} = t_{\delta_j} t_{\varepsilon_i}$ for $j \neq 2i$,

- (R4) $t_{\delta_i} t_{\delta_{i+1}} t_{\delta_i} = t_{\delta_{i+1}} t_{\delta_i} t_{\delta_{i+1}}$ for $1 \leq i \leq g-2$,
 (R5) $t_{\varepsilon_i} t_{\delta_{2i}} t_{\varepsilon_i} = t_{\delta_{2i}} t_{\varepsilon_i} t_{\delta_{2i}}$ for $2i < g$;

The relations involving crosscap transpositions are not so well known and we refer the reader to [18] and [22] for their proofs.

- (R6) $t_{\delta_i} u_j = u_j t_{\delta_i}$ for $|i-j| > 1$,
 (R7) $u_i u_j = u_j u_i$ for $|i-j| > 1$,
 (R8) $t_{\varepsilon_i} u_j = u_j t_{\varepsilon_i}$ for $j > 2i$,
 (R9) $u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1}$ for $1 \leq i \leq g-2$,
 (R10) $t_{\delta_i} u_{i+1} u_i = u_{i+1} u_i t_{\delta_{i+1}}$ for $1 \leq i \leq g-2$,
 (R11) $u_{i+1} t_{\delta_i} t_{\delta_{i+1}} u_i = t_{\delta_i} t_{\delta_{i+1}}$ for $1 \leq i \leq g-2$;
 (R12) $t_{\delta_i} u_i t_{\delta_i} = u_i$ for $1 \leq i \leq g-1$.

It follows from (R4) that all t_{δ_i} are conjugate for $1 \leq i \leq g-1$, by (R5) t_{ε_j} is conjugate to $t_{\delta_{2j}}$ for $2j < g$, and by (R12) t_{δ_i} is conjugate to $t_{\delta_i}^{-1}$. Similarly, by (R9) all u_i are conjugate for $1 \leq i \leq g-1$, and by (R11) u_i is conjugate to u_i^{-1} .

For a group G we denote the abelianization $G/[G, G]$ by G^{ab} . The following theorem is proved in [12] for $n = 0$ and generalised to $n > 0$ in [19].

Theorem 3.2. *For $n \leq 1$ and $g \geq 3$, $\mathcal{M}(N_{g,n})^{\text{ab}}$ has the following presentation as a \mathbb{Z} -module.*

$$\begin{aligned} & \langle [t_{\delta_1}], [t_{\varepsilon_2}], [u_1] \mid 2[t_{\delta_1}] = 2[t_{\varepsilon_2}] = 2[u_1] = 0 \rangle \quad \text{if } g = 4, \\ & \langle [t_{\delta_1}], [u_1] \mid 2[t_{\delta_1}] = 2[u_1] = 0 \rangle \quad \text{if } g \in \{3, 5, 6\}, \\ & \langle [u_1] \mid 2[u_1] = 0 \rangle \quad \text{if } g \geq 7. \end{aligned}$$

In particular, for $g \geq 7$ we have $[t_{\delta_1}] = 0$.

Lemma 3.3. *For $g \geq 5$ and $n \leq 1$ let α, β be two-sided curves on $N_{g,n}$, intersecting transversally in one point. If $f: \mathcal{M}(N_{g,n}) \rightarrow G$ is a homomorphism, such that $f(t_\alpha)$ commutes with $f(t_\beta)$, then $\text{Im}(f)$ is abelian.*

Proof. Let $N = N_{g,n}$ and $\mathcal{M} = \mathcal{M}(N_{g,n})$. Fix a regular neighbourhood A of $\alpha \cup \beta$. Note that A is homeomorphic to $S_{1,1}$ and $N \setminus A$ is homeomorphic to $N_{g-2,1}$. It follows that for each $i \leq g-2$ there is a homeomorphism $h: N \rightarrow N$ such that $h(\alpha) = \delta_i$ and $h(\beta) = \delta_{i+1}$. It follows that $ht_\alpha h^{-1} = t_{\delta_i}^{\varepsilon_1}$ and $ht_\beta h^{-1} = t_{\delta_{i+1}}^{\varepsilon_2}$, where $\varepsilon_j \in \{-1, 1\}$ for $j = 1, 2$. Hence $f(t_{\delta_i})$ commutes with $f(t_{\delta_{i+1}})$ and by the braid relation (R4) $f(t_{\delta_i}) = f(t_{\delta_{i+1}})$. Analogously, $f(t_{\varepsilon_2}) = f(t_{\delta_4})$. By Theorem 3.1, $\text{Im}(f)$ is generated by $f(t_{\delta_1})$ and $f(u_{g-1})$, and since u_{g-1} commutes with t_{δ_1} , thus $\text{Im}(f)$ is abelian. \square

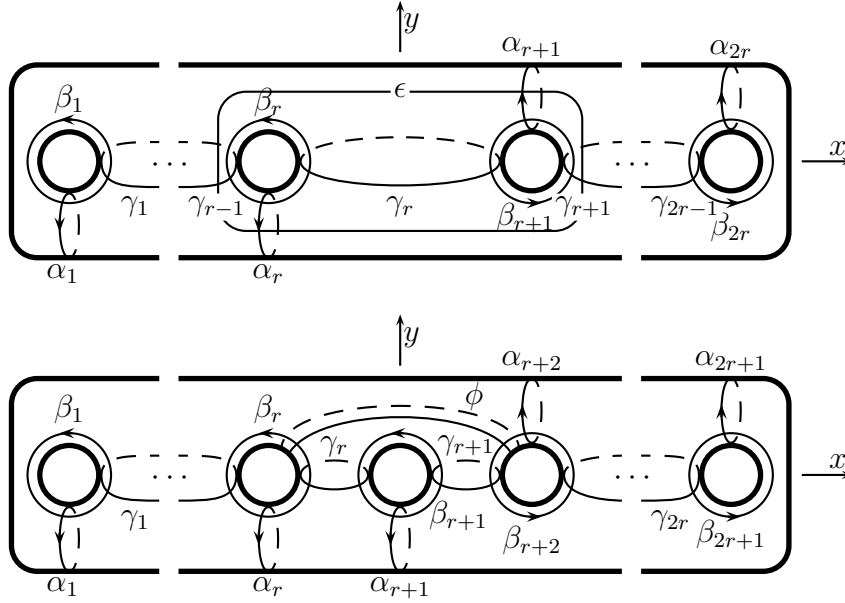


FIGURE 3. The surface S_{g-1} for $g = 2r + 1$ (top) and $g = 2r + 2$ (bottom).

Lemma 3.4. *Suppose that $g \geq 4$ and $f: \mathcal{M}(N_{g,n}) \rightarrow G$ is a homomorphism. If $f(t_{\varepsilon_i}) = f(t_{\delta_j})$ for some $2i + 1 \leq j \leq g - 1$, then $f(t_{\delta_1}^2) = 1$.*

Proof. Set $x = f(t_{\varepsilon_i}) = f(t_{\delta_j})$ and $y = f(u_j)$. By the relation (R8) we have $xy = yx$, and by (R12) $xyx = y$. Hence $x^2 = 1$ which finishes the proof, because t_{δ_j} is conjugate to t_{δ_1} . \square

Let $g = 2r + s$, where $r \geq 1$, $s \in \{1, 2\}$ and $S = S_{g-1}$. Consider S as being embedded in \mathbb{R}^3 in such a way that it is invariant under the reflections about the xy , xz and yz planes, as shown on Figure 3. We define a homeomorphism $j: S \rightarrow S$ as $j(x, y, z) = (-x, -y, -z)$. The quotient space S/j is a nonorientable surface of genus g and the projection $p: S \rightarrow S/j$ is a covering map of degree 2. Let S' be the subsurface of S consisting of points $(x, y, z) \in S$ with $x \leq -\varepsilon$, where ε is a positive constant, so small that S' is homeomorphic to $S_{r,s}$. If g is even, then one of the boundary components of S' is isotopic to α_{r+1} . In this paper we identify isotopic curves, and therefore we will treat α_{r+1} as a curve on S' . Note that the restriction of p to S' is an embedding. For odd g we define γ' to be the arc of γ_r consisting of points with $x \leq 0$. For even g we define β' to be the arc of β_{r+1} consisting of points with $x \leq 0$. Note that $p(\gamma')$ and $p(\beta')$ are one-sided simple closed curves on S/j .

Proposition 3.5. *There is a homeomorphism $\varphi: S_{g-1}/j \rightarrow N_g$ such that, for $P = \varphi \circ p$, up to isotopy*

- (1) $P(\beta_i) = \delta_{2i}$ for $1 \leq i \leq r$,
- (2) $P(\alpha_i) = \varepsilon_i$ for $2 \leq 2i \leq g$,
- (3) $P(\gamma_i) = \delta_{2i+1}$ for $2 \leq 2i \leq g-2$,
- (4) $P(\gamma') = \xi_g$ if g is odd,
- (5) $P(\beta') = \xi_g$ if g is even.

Proof. Observe that the curves δ_i for $1 \leq i \leq g-1$ form a chain of two-sided curves, which means that δ_i and δ_j intersect at one point if $|i-j| = 1$, and they are disjoint otherwise. It follows that a regular neighbourhood of the union of δ_i for $1 \leq i \leq g-1$ is homeomorphic to $S_{r,s}$. Let Σ be such a neighbourhood, which may be taken to contain the curves ε_i for $2 \leq 2i \leq g$ (if g is even, then one of the boundary components of Σ is isotopic to ε_{r+1}). Note that ε_i , ε_{i+1} and δ_{2i+1} bound a pair of pants for $2 \leq 2i \leq g-2$. It follows that there exists a homeomorphism $\varphi: S_{g-1}/j \rightarrow N_g$ such that, for $P = \varphi \circ p$, we have $P(S') = \Sigma$ and the conditions (1, 2, 3) are satisfied. Observe that $N_g \setminus \Sigma$ is a Möbius strip (if g is odd) or an annulus (if g is even), whose core (isotopic to $\xi_{\{1, \dots, g\}}$) intersects ξ_g once. By looking at the intersection of ξ_g with the curves δ_i , ε_j it is easy to see that φ can be taken to satisfy also the condition (4) or (5). \square

Corollary 3.6. *There is a homomorphism $\iota: \mathcal{M}(S') \rightarrow \mathcal{M}(N_{g,n})$ such that*

- $\iota(t_{\beta_i}) = t_{\delta_{2i}}$ for $1 \leq i \leq r$,
- $\iota(t_{\alpha_i}) = t_{\varepsilon_i}$ for $2 \leq 2i \leq g$,
- $\iota(t_{\gamma_i}) = t_{\delta_{2i+1}}$ for $2 \leq 2i \leq g-2$,

where the Dehn twists about the curves on S' are right with respect to the standard orientation.

Proof. By the proof of Proposition 3.5, the restriction of P to S' is a homeomorphism onto Σ satisfying the conditions (1,2,3). There is an induced isomorphism $\mathcal{M}(S') \rightarrow \mathcal{M}(\Sigma)$, which may be composed with the homomorphism $\mathcal{M}(\Sigma) \rightarrow \mathcal{M}(N_{g,n})$ induced by the inclusion $\Sigma \hookrightarrow N_{g,n}$, for any $n \geq 0$, to obtain ι . \square

For any homeomorphism $h: N_g \rightarrow N_g$ there is a unique orientation preserving lift $\tilde{h}: S_{g-1} \rightarrow S_{g-1}$ such that $h \circ P = P \circ \tilde{h}$. By [3], the mapping $h \mapsto \tilde{h}$ induces a monomorphism $\theta: \mathcal{M}(N_g) \rightarrow \mathcal{M}(S_{g-1})$. The following proposition follows from [3] and [22, Theorem 10], where the lift of a crosscap transposition is determined.

Proposition 3.7. *There is a monomorphism $\theta: \mathcal{M}(N_g) \rightarrow \mathcal{M}(S_{g-1})$ such that*

$$\theta(t_{\varepsilon_i}) = t_{\alpha_i} t_{\alpha_{g-i}}^{-1}, \quad \theta(t_{\delta_{2i}}) = t_{\beta_i} t_{\beta_{g-i}}^{-1}, \quad \theta(t_{\delta_{2j+1}}) = t_{\gamma_j} t_{\gamma_{g-1-j}}^{-1},$$

for $1 \leq i \leq r$, $2 \leq 2j \leq g-2$ and

$$\theta(u_{g-1}) = \begin{cases} t_{\beta_r}^{-1} t_{\beta_{r+1}} (t_{\gamma_r} t_{\beta_r} t_{\beta_{r+1}})^2 t_{\epsilon}^{-1} & \text{if } g = 2r + 1, \\ t_{\gamma_r}^{-1} t_{\gamma_{r+1}} (t_{\beta_{r+1}} t_{\gamma_r} t_{\gamma_{r+1}})^2 t_{\phi}^{-1} & \text{if } g = 2r + 2. \end{cases}$$

4. HOMOLOGICAL REPRESENTATIONS

Fix $g \geq 3$ and let $S = S_{g-1}$, $N = N_g$ and $P: S \rightarrow N$ be as in the previous section. The group $H_1(S)$ is a free \mathbb{Z} -module of rank $2(g-1)$ and the homology classes $a_i = [\alpha_i]$, $b_i = [\beta_i]$ for $1 \leq i \leq g-1$ form its basis, which is a symplectic basis with respect to the algebraic intersection form:

$$\langle a_i, a_j \rangle = 0, \quad \langle b_i, b_j \rangle = 0, \quad \langle a_i, b_j \rangle = \delta_{ij}.$$

Let $\Phi: \mathcal{M}(S) \rightarrow \text{Sp}(H_1(S))$ be the homomorphism induced by the action of $\mathcal{M}(S)$ on $H_1(S)$. If γ is an oriented simple closed curve on S , $[\gamma] \in H_1(S)$ is its homology class, and t_γ is the right Dehn twist, then $\Phi(t_\gamma)$ is the transvection

$$(4.1) \quad \Phi(t_\gamma)(h) = h + \langle [\gamma], h \rangle [\gamma], \quad \text{for } h \in H_1(S).$$

From this formula we immediately obtain that, with respect to the basis $(a_1, b_1, \dots, a_{g-1}, b_{g-1})$, we have

$$\Phi(t_{\alpha_i}) = A_i, \quad \Phi(t_{\beta_i}) = B_i, \quad \Phi(t_{\gamma_j}) = C_j,$$

for $1 \leq i \leq g-1$, $1 \leq j \leq g-2$, where A_i , B_i and C_j are the matrices defined in Section 2.

The group $H_1(N)$ has the following presentation, as a \mathbb{Z} -module:

$$H_1(N) = \langle x_1, \dots, x_g \mid 2(x_1 + \dots + x_g) = 0 \rangle,$$

where $x_i = [\xi_i]$. Set $k = x_1 + \dots + x_g$ and $R = H_1(N)/\langle k \rangle$. Observe that k is the unique element of order two in $H_1(N)$ and R is a free \mathbb{Z} -module of rank $g-1$.

The map $P: S \rightarrow N$ induces $P_*: H_1(S) \rightarrow H_1(N)$, such that, for $1 \leq i \leq r$

$$\begin{aligned} P_*(a_i) &= x_1 + \dots + x_{2i} = -P_*(a_{g-i}), \\ P_*(b_i) &= x_{2i} + x_{2i+1} = P_*(b_{g-i}), \end{aligned}$$

and if $g = 2r + 2$ then

$$P_*(a_{r+1}) = x_1 + \dots + x_g = k, \quad P_*(b_{r+1}) = 2x_g.$$

Let $q: H_1(S) \rightarrow R$ be the composition of P_* with the canonical projection $H_1(N) \rightarrow R$, and set $K = \ker q$. It is easy to verify that K has rank $g - 1$ and the following elements form its basis:

$$\begin{aligned} e_i &= a_i + a_{g-i}, & e_{r+i} &= b_i - b_{g-i} & \text{for } 1 \leq i \leq r, \\ e_{2r+1} &= a_{r+1} & & & \text{for } g = 2r + 2. \end{aligned}$$

We also set

$$\begin{aligned} f_i &= b_i, & f_{r+i} &= a_{g-i} & \text{for } 1 \leq i \leq r, \\ f_{2r+1} &= b_{r+1} & & & \text{for } g = 2r + 2. \end{aligned}$$

Observe that the elements e_i, f_i for $1 \leq i \leq g - 1$ form a symplectic basis of $H_1(S)$. It follows that $H_1(S)/K$ is a free \mathbb{Z} -module of rank $g - 1$, which is canonically isomorphic to R if g is odd, or to an index-two subgroup of R if g is even. The group $\mathcal{M}(N)$ acts on $H_1(S)$ by the composition $\Phi \circ \theta: \mathcal{M}(N) \rightarrow \text{Sp}(H_1(S))$. Observe that K is $\mathcal{M}(N)$ -invariant and hence we have two $(g - 1)$ -dimensional representations

$$\psi_1: \mathcal{M}(N) \rightarrow \text{GL}(K), \quad \psi_2: \mathcal{M}(N) \rightarrow \text{GL}(H_1(S)/K).$$

Lemma 4.1. $\ker \Psi_1 = \ker \Psi_2$ and $\theta(\ker \Psi_1) \subset \ker \Phi$.

Proof. Fix the basis $(e_1, \dots, e_{g-1}, f_1, \dots, f_{g-1})$ of $H_1(S)$. For any $x \in \mathcal{M}(N)$ let X be the matrix of $\Phi(\theta(x))$. We have $X = \begin{pmatrix} X_1 & Y \\ 0 & X_2 \end{pmatrix}$, where X_1, X_2, Y are $(g - 1) \times (g - 1)$ matrices. The matrix of the algebraic intersection form is $\Omega = \begin{pmatrix} 0 & I_{g-1} \\ -I_{g-1} & 0 \end{pmatrix}$ and since X is symplectic, we have $X^t \Omega X = \Omega$, which gives $X_1^t X_2 = I$. Therefore $X_1 = I \Leftrightarrow X_2 = I$, which proves $\ker \Psi_1 = \ker \Psi_2$. To prove the second part of the lemma, assume $X_1 = X_2 = I$. Let $j_*: \mathcal{M}(S) \rightarrow \mathcal{M}(S)$ be the map induced by the covering involution j . It is easy to check that the matrix of j_* has the form $J = \begin{pmatrix} -I_{g-1} & T \\ 0 & I_{g-1} \end{pmatrix}$ for some T . We have $XJ = JX$, which implies $Y = 0$. \square

Note that $\ker \Phi$ is the Torelli group, which is well known to be torsion free, and since θ is a monomorphism, we immediately obtain the following.

Corollary 4.2. $\ker \Psi_1$ is torsion free. \square

Remark 4.3. Let H denote the subgroup of $\mathcal{M}(N)$ consisting of the elements inducing the identity on $H_1(N)$. It was proved in [9] that $\theta(H) \subset \ker \Phi$. We leave it as an exercise to check that if g is odd, then $H = \ker \Psi_2$, whereas if g is even, then H is an index-two subgroup

of $\ker \Psi_2$. In the latter case, if $g = 2r + 2$, then we have $\ker \Psi_2 = H \cup t_{\varepsilon_{r+1}} H$.

Remark 4.4. There is a nontrivial action of $\pi_1(N)$ on \mathbb{Z} defined as follows: $\gamma \in \pi_1(N)$ acts by multiplication by 1 or -1 according to whether γ preserves or reverses local orientations of N . This action gives rise to homology groups with local coefficients $H_*(N, \tilde{\mathbb{Z}})$, where $\tilde{\mathbb{Z}}$ is \mathbb{Z} with the nontrivial $\mathbb{Z}[\pi_1(N)]$ -module structure. By [11, Example 3H.3], we have the exact sequence

$$H_2(N) \rightarrow H_1(N, \tilde{\mathbb{Z}}) \rightarrow H_1(S) \xrightarrow{P_*} H_1(N),$$

which is a part of a long exact sequence of homology groups. Since $H_2(N) = 0$, we have a $\mathcal{M}(N)$ -equivariant isomorphism $H_1(N, \tilde{\mathbb{Z}}) \cong \ker P_*$. If g is odd, then $\ker P_* = K$, whereas if g is even, then $\ker P_*$ is an index-two subgroup of K . Therefore the representations Ψ_1 and Ψ_2 may be seen as coming from the actions of $\mathcal{M}(N)$ on $H_1(N, \tilde{\mathbb{Z}})$ and $H_1(N)$ respectively.

For K we fix the basis

$$\begin{aligned} (e_1, e_{r+1}, \dots, e_r, e_{2r}) & \text{ if } g = 2r + 1, \\ (e_1, e_{r+1}, \dots, e_r, e_{2r}, e_{2r+1}) & \text{ if } g = 2r + 2. \end{aligned}$$

For $H_1(S)/K$ we fix the basis

$$\begin{aligned} (a_1 + K, b_1 + K, \dots, a_r + K, b_r + K) & \text{ if } g = 2r + 1, \\ (a_1 + K, b_1 + K, \dots, a_r + K, b_r + K, b_{r+1} + K) & \text{ if } g = 2r + 2. \end{aligned}$$

Having fixed bases for K and $H_1(S)/K$ we can now compute, for Ψ_1 and Ψ_2 , the images of the generators of $\mathcal{M}(N)$. This is done by a straightforward calculation, using Proposition 3.7 and the formula (4.1). For $k = 1, 2$ and $1 \leq i \leq r$, $1 \leq j \leq r - 1$ we have

$$\Psi_k(t_{\varepsilon_i}) = A_i, \quad \Psi_k(t_{\delta_{2i}}) = B_i, \quad \Psi_k(t_{\delta_{2j+1}}) = C_j.$$

If $g = 2r + 1$ then

$$\Psi_1(u_{g-1}) = \begin{pmatrix} I_{g-3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}, \quad \Psi_2(u_{g-1}) = \begin{pmatrix} I_{g-3} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

If $g = 2r + 2$ then

$$\Psi_1(t_{\delta_{g-1}}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}, \quad \Psi_2(t_{\delta_{g-1}}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_1(u_{g-1}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 \end{pmatrix}, \quad \Psi_2(u_{g-1}) = \begin{pmatrix} I_{g-4} & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Now it is easy to see that Ψ_1 and Ψ_2 are not conjugate as homomorphism to $\mathrm{GL}(g-1, \mathbb{C})$. For suppose that there is $M \in \mathrm{GL}(g-1, \mathbb{C})$, such that $\Psi_1(x) = M\Psi_2(x)M^{-1}$ for all $x \in \mathcal{M}(N)$. Then M commutes with A_i, B_i, C_j for $1 \leq i \leq r, 1 \leq j \leq r-1$, and by Lemma 2.1, $M = \alpha I_{2r}$ if $g = 2r+1$, or $M = \mathrm{diag}(\alpha I_{2r}, \beta)$ if $g = 2r+2$, for $\alpha, \beta \in \mathbb{C}$. In either case it is impossible that $\Psi_1(u_{g-1}) = M\Psi_2(u_{g-1})M^{-1}$.

5. HOMOMORPHISMS FROM $\mathcal{M}(N_{g,n})$ TO $\mathrm{GL}(m, \mathbb{C})$ FOR $m < g-1$

The aim of this section is to prove Theorem 1.3. The proof is divided in two parts.

Proof of Theorem 1.3 for $(g, m) \neq (6, 4)$. Suppose that $n \in \{0, 1\}$, $g = 2r + s$ for $r \geq 2, s \in \{1, 2\}$, $m \leq g-2$ and $f: \mathcal{M}(N_{g,n}) \rightarrow \mathrm{GL}(m, \mathbb{C})$ is a homomorphism. By Theorem 3.2, it suffices to prove that $\mathrm{Im}(f)$ is abelian. Let $S' = S_{r,s}$ and $\iota: \mathcal{M}(S') \rightarrow \mathcal{M}(N_{g,n})$ be the homomorphism from Corollary 3.6. Set $f' = f \circ \iota$ and observe that if $\mathrm{Im}(f')$ is abelian, then so is $\mathrm{Im}(f)$, by Lemma 3.3.

Suppose that $m \leq 2r-1$. Then $\mathrm{Im}(f')$ is either trivial or cyclic by Theorem 1.1 and we are done. This finishes the proof for odd g .

Suppose that $g = 2r+2$ for $r \geq 3$ and $m = 2r$. By Theorem 1.2, f' is either trivial or conjugate to the homological representation Φ . In the former case we are done. In the latter case, by the definition of Φ we have $\Phi(t_{\gamma_r}) = \Phi(t_{\alpha_r})$ because the curves γ_r and α_r become isotopic after gluing discs to the boundary of S' . It follows that $f(t_{\delta_{2r+1}}) = f(t_{\varepsilon_r})$ and by Lemma 3.4 $f(t_{\delta_1}^2) = 1$. This is a contradiction because $\Phi(t_{\alpha_1})$ has infinite order. \square

In order to prove Theorem 1.3 for $(g, m) = (6, 4)$, we first prove some lemmas.

Lemma 5.1. *Suppose that $f: \mathcal{M}(N_{4,n}) \rightarrow \mathrm{GL}(2, \mathbb{C})$ is a homomorphism. Then, with respect to some basis one of the following cases holds.*

- (1) $f(t_{\delta_1}) = f(t_{\delta_2}) = f(t_{\delta_3}) = \lambda I, \lambda \in \{-1, 1\}$
- (2) $f(t_{\delta_1}) = f(t_{\delta_2}) = f(t_{\delta_3}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
- (3) $f(t_{\delta_1}) = f(t_{\delta_3}) = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, f(t_{\delta_2}) = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}.$

In particular $f(t_{\delta_1}^2) = 1$.

Proof. For $i = 1, 2, 3$ let $L_i = f(t_{\delta_i})$ and $U = f(u_3)$.

Suppose that L_1 has only one eigenvalue λ . Since L_1 is conjugate to L_1^{-1} (by (R12)), we have $\lambda \in \{-1, 1\}$. If $\dim E(L_1, \lambda) = 2$, then we have the case (1). Suppose that $\dim E(L_1, \lambda) = 1$. If $E(L_1, \lambda) \neq E(L_2, \lambda)$, then with respect to some basis we have $L_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $L_2 = \begin{pmatrix} \lambda & 0 \\ x & \lambda \end{pmatrix}$ for some x , and from the braid relation $L_1 L_2 L_1 = L_2 L_1 L_2$ we have $x = -1$. Since L_3 commutes with L_1 we have $L_3 = \begin{pmatrix} \lambda & y \\ 0 & \lambda \end{pmatrix}$ for some y , and from $L_2 L_3 L_2 = L_3 L_2 L_3$ we obtain $y = 1$, hence $L_1 = L_3$. Since $\delta_1 = \varepsilon_1$, we have $L_1^2 = I$ by Lemma 3.4 (for $i = 1, j = 3$), which is a contradiction. If $E(L_1, \lambda) = E(L_2, \lambda)$, then with respect to some basis we have $L_1 = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $L_2 = \begin{pmatrix} \lambda & x \\ 0 & \lambda \end{pmatrix}$, and it is easy to obtain a contradiction as above, by showing that $L_1 = L_2 = L_3$.

Suppose that L_1 has two eigenvalues λ, μ . Then with respect to some basis we have $L_1 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, and since L_3 and U commute with L_1 , they are also diagonal. In particular we have $UL_3 = L_3U$ and $L_3UL_3 = U$ (R12) gives $L_3^2 = 1$, which implies $\{\lambda, \mu\} = \{-1, 1\}$. We have $L_3 = L_1$ or $L_3 = -L_1$. In the latter case the braid relations $L_3 L_2 L_3 = L_2 L_3 L_2$ and $L_1 L_2 L_1 = L_2 L_1 L_2$ imply $L_2 L_1 L_2 = 0$, a contradiction, hence $L_1 = L_3$.

If $E(L_1, 1) \neq E(L_2, 1)$, then with respect to some basis we have $L_1 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$, $L_2 = \begin{pmatrix} -1 & 0 \\ x & 1 \end{pmatrix}$. From $L_1 L_2 L_1 = L_2 L_1 L_2$ we have $x = 1$ and we are in the case (3). Analogously, if $E(L_1, -1) \neq E(L_2, -1)$, then with respect to some basis we have $L_1 = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}$, $L_2 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$, and since $E(L_1, 1) \neq E(L_1, 1)$, we are in the case (3) again.

Finally, if $E(L_1, 1) = E(L_2, 1)$ and $E(L_1, -1) = E(L_2, -1)$, then with respect to some basis we have $L_1 = L_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and we are in the case (2). \square

Lemma 5.2. *Suppose that $n \leq 1$ and $f: \mathcal{M}(N_{6,n}) \rightarrow \mathrm{GL}(4, \mathbb{C})$ is a homomorphism such that $f(t_{\delta_1}^2) = 1$. Then $\mathrm{Im}(f)$ is abelian.*

Proof. Let H be the normal closure of $t_{\delta_1}^2$ in $\mathcal{M}(N_{6,n})$ and set $G = \mathcal{M}(N_{6,n})/H$. We have an induced homomorphism $f': G \rightarrow \mathrm{GL}(4, \mathbb{C})$

such that $f = f' \circ \pi$, where $\pi: \mathcal{M}(N_{6,n}) \rightarrow G$ is the canonical projection. By the relations (R1, R4), the mapping $\rho(\sigma_i) = \pi(t_{\delta_i})$, where σ_i is the transposition $(i, i+1)$ for $1 \leq i \leq 5$, defines a homomorphism $\rho: \mathfrak{S}_6 \rightarrow G$. Let $\phi: \mathfrak{S}_6 \rightarrow \mathrm{GL}(4, \mathbb{C})$ be the composition $f' \circ \rho$. By Lemma 2.2, ϕ is the direct sum of one-dimensional representations. In particular the image of ϕ is abelian, and so is $\mathrm{Im}(f)$ by Lemma 3.3. \square

Let R be the subsurface obtained by removing from $N_{6,n}$ a regular neighbourhood of $\delta_1 \cup \delta_2$. Note that R is homeomorphic to $N_{4,n+1}$. The homomorphism $\mathcal{M}(R) \rightarrow \mathcal{M}(N_{6,n})$ induced by the inclusion of R in $N_{6,n}$ is injective, and we will treat $\mathcal{M}(R)$ as a subgroup of $\mathcal{M}(N_{6,n})$.

Lemma 5.3. *Suppose that $n \leq 1$, $f: \mathcal{M}(N_{6,n}) \rightarrow \mathrm{GL}(4, \mathbb{C})$ is a homomorphism and there exists a splitting $\mathbb{C}^4 = V_1 \oplus V_2$ such that V_i is a 2-dimensional $\mathcal{M}(R)$ -invariant subspace for $i = 1, 2$. Then $\mathrm{Im}(f)$ is abelian.*

Proof. Let f' be the restriction of f to $\mathcal{M}(R)$. With respect to the splitting $\mathbb{C}^4 = V_1 \oplus V_2$ we have $f' = f_1 \oplus f_2$ for some $f_i: \mathcal{M}(R) \rightarrow \mathrm{GL}(2, \mathbb{C})$, $i = 1, 2$. By Lemma 5.1 we have $f_i(t_{\delta_4}^2) = 1$ for $i = 1, 2$, hence $f(t_{\delta_4}^2) = 1$ and we are done by Lemma 5.2. \square

Lemma 5.4. *Suppose that $n \leq 1$, $f: \mathcal{M}(N_{6,n}) \rightarrow \mathrm{GL}(4, \mathbb{C})$ is a homomorphism, $f(t_{\delta_1})$ has only one eigenvalue and there exists a 2-dimensional $\mathcal{M}(R)$ -invariant subspace. Then $\mathrm{Im} f$ is abelian.*

Proof. Let λ be the eigenvalue of $f(t_{\delta_1})$. Fix a basis of \mathbb{C}^4 whose first two vectors span the $\mathcal{M}(R)$ -invariant subspace. By the case (1) of Lemma 5.1, with respect to such basis we have $f(t_{\delta_4}) = \begin{pmatrix} \lambda I & X \\ 0 & \lambda I \end{pmatrix}$,

$f(t_{\delta_5}) = \begin{pmatrix} \lambda I & Y \\ 0 & \lambda I \end{pmatrix}$, for some 2×2 matrices X, Y . In particular $f(t_{\delta_4})$ and $f(t_{\delta_5})$ commute and we are done by Lemma 3.3. \square

Proof of Theorem 1.3 for $g = 6$, $m = 4$. Suppose that $n \in \{0, 1\}$ and $f: \mathcal{M}(N_{6,n}) \rightarrow \mathrm{GL}(4, \mathbb{C})$ is a homomorphism. For $1 \leq i \leq 5$ we set $L_i = f(t_{\delta_i})$ and $M = f(t_{\varepsilon_2})$, $U_5 = f(u_5)$. We consider the following cases.

- (1) L_1 has 4 eigenvalues.
- (2) L_1 has 3 eigenvalues.
- (3) L_1 has 2 eigenvalues with equal multiplicities.
- (4) L_1 has 2 eigenvalues with different multiplicities.
- (5) L_1 has 1 eigenvalue.

In the cases (1, 2, 3) it is easy to find a splitting $\mathbb{C}^4 = V_1 \oplus V_2$ such that V_i is a 2-dimensional $\mathcal{M}(R)$ -invariant subspace for $i = 1, 2$. For

example, suppose that L_1 has three eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that $\#\lambda_1 = \#\lambda_2 = 1$ and $\#\lambda_3 = 2$. Then we take $V_1 = E(L_1, \lambda_1) \oplus E(L_1, \lambda_2)$ and $V_2 = E(L_1, \lambda_3)$ if $\dim E(L_1, \lambda_3) = 2$ or $V_2 = E^2(L_1, \lambda_3)$ if $\dim E(L_1, \lambda_3) = 1$. Therefore in the cases (1, 2, 3) we are done by Lemma 5.3.

Assume (5). Let λ be the unique eigenvalue of L_1 and $k = \dim E(L_1, \lambda)$. If $k = 4$ then $L_1 = \lambda I$ and the image of f is cyclic. If $k = 2$ or $k = 1$ then respectively $E(L_1, \lambda)$ or $E^2(L_1, \lambda)$ is a 2-dimensional $\mathcal{M}(R)$ -invariant subspace, and we are done by Lemma 5.4. Suppose that $k = 3$. If $E(L_1, \lambda) \neq E(L_2, \lambda)$ then $E(L_1, \lambda) \cap E(L_2, \lambda)$ is a 2-dimensional $\mathcal{M}(R)$ -invariant subspace, and we are done by Lemma 5.4. If $E(L_1, \lambda) = E(L_2, \lambda)$ then with respect to some basis we have

$$L_1 = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}, \quad L_2 = \begin{pmatrix} \lambda & 0 & 0 & x \\ 0 & \lambda & 0 & y \\ 0 & 0 & \lambda & z \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

In particular L_1 and L_2 commute and we are done by Lemma 3.3.

It remains to consider the case (4). Suppose that L_1 has eigenvalues μ, λ , with $\#\mu = 1$ and $\#\lambda = 3$. Since L_1 is conjugate to L_1^{-1} , we have $\{\mu, \lambda\} = \{-1, 1\}$. It follows from Theorem 3.2 that there is a homomorphism $\tau(\mathcal{M}(N_6)) \rightarrow \{-1, 1\}$ such that $\tau(a_1) = -1$. By multiplying f by τ if necessary, we may assume $\mu = -1, \lambda = 1$. The Jordan form of L_1 is one of the following three matrices.

$$(i) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (ii) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (iii) \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

In the case (i) we have $L_1^2 = I$ and we are done by Lemma 5.2.

In the case (ii) the following subspaces are $\mathcal{M}(R)$ -invariant: $E(L_1, -1)$, $E(L_1, 1)$, $E^2(L_1, 1)$, $E^3(L_1, 1)$. It follows that

$$M = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & v_1 & v_2 \\ 0 & 0 & x_3 & v_3 \\ 0 & 0 & 0 & x_4 \end{pmatrix}, \quad L_4 = \begin{pmatrix} y_1 & 0 & 0 & 0 \\ 0 & y_2 & w_1 & w_2 \\ 0 & 0 & y_3 & w_3 \\ 0 & 0 & 0 & y_4 \end{pmatrix}.$$

The braid relation $ML_4M = L_4ML_4$ (R5) implies $x_i = y_i$ for $1 \leq i \leq 4$. Since the first two vectors of the basis are eigenvectors of M , they have to correspond to different eigenvalues of M . Therefore $x_2 = -x_1$, $x_3 = x_4 = 1$ and $x_1 = 1$ or $x_1 = -1$. In either case it is not difficult

to check that $ML_4M = L_4ML_4$ holds if and only if $M = L_4$. We are done by Lemma 3.3.

In the case (iii) the following subspaces are $\mathcal{M}(R)$ -invariant: $E(L_1, -1)$, $E(L_1, 1)$, $E^2(L_1, 1)$. We have $\dim E(L_1, 1) = 2$ and by applying Lemma 5.1 to the action of $\mathcal{M}(R)$ on this subspace, we obtain three sub-cases.

Sub-case (iiia). If the action of $\mathcal{M}(R)$ on $E(L_1, 1)$ is trivial, then we have

$$M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As in the case (ii), the braid relation implies $M = L_4$ and we are done by Lemma 3.3.

Sub-case (iiib). By changing the basis of $E(L_1, 1)$ we may assume that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & x_1 \\ 0 & 0 & 1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

As in the case (ii), the braid relation implies $M = L_4$ and we are done by Lemma 3.3.

Sub-case (iiic). By changing the basis of $E(L_1, 1)$ we may assume that

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & x_1 \\ 0 & 0 & -1 & x_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & y_1 \\ 0 & 1 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad L_5 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & z_1 \\ 0 & 0 & -1 & z_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By solving the equations $ML_4M = L_4ML_4$ and $L_5L_4L_5 = L_4L_5L_4$ we obtain $x_2 = -(2x_1 + y_1 + 2y_2)$, $z_2 = -(2z_1 + y_1 + 2y_2)$, and from $ML_5 = L_5M$ we obtain $x_2 = z_2$. Thus $M = L_5$ and by Lemma 3.4 $L_1^2 = 1$. We are done by Lemma 5.2. \square

6. HOMOMORPHISMS BETWEEN MAPPING CLASS GROUPS

The aim of this section is to prove Theorem 1.4. Fix $g \geq 5$ and set $\mathcal{M} = \mathcal{M}(N_g)$. We are going to use the fact that $s = t_{\delta_1} \cdots t_{\delta_{g-1}}$ has finite order in \mathcal{M} (equal to g if it is even, or $2g$ otherwise, see [18]). By the relations (R1,R4) we have

$$(6.1) \quad t_{\delta_{i+1}}s = st_{\delta_i} \quad \text{for } 1 \leq i \leq g-2.$$

By Theorem 3.2 we have $s \in [\mathcal{M}, \mathcal{M}]$ for $g \geq 7$ and $g = 5$, $s^2 \in [\mathcal{M}, \mathcal{M}]$ for $g = 6$.

Proof of Theorem 1.4. Suppose that $g \geq 5$, $h < g$ and $f: \mathcal{M}(N_g) \rightarrow \mathcal{M}(N_h)$ is a homomorphism. Since $M(N_h)$ is abelian for $h \leq 2$, we are assuming $h \geq 3$.

Let $f': \mathcal{M}(N_g) \rightarrow \mathrm{GL}(h-1, \mathbb{C})$ be the composition $\Psi_1 \circ f$ and $K = \ker \Psi_1$. By Theorem 1.3, $\mathrm{Im}(f')$ is abelian, hence $f([\mathcal{M}(N_g), \mathcal{M}(N_g)]) \subseteq K$. Suppose that $g \geq 7$ or $g = 5$. Then $f(s) \in K$, and since K is torsion free by Lemma 4.2, thus $f(s) = 1$. This gives, by (6.1), $f(t_{\delta_1}) = f(t_{\delta_2})$ and we are done by Lemma 3.3. If $g = 6$ then $f(s^2) \in K$, which gives $f(s^2) = 1$ and $f(t_{\delta_2}) = f(t_{\delta_4})$. Since t_{δ_1} commutes with t_{δ_4} , thus $f(t_{\delta_1})$ commutes with $f(t_{\delta_2})$ and we are done by Lemma 3.3. \square

Note that Theorems 1.3 and 1.4 are trivially true for $g \leq 3$ because $\mathrm{GL}(1, \mathbb{C}) = \mathbb{C}^*$, $\mathcal{M}(N_2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $M(N_1) = 1$ are abelian groups. On the other hand, Corollary 6.2 below shows that both theorems are false for $g = 4$ (recall that $\mathcal{M}(N_3) \cong \mathrm{GL}(2, \mathbb{Z})$). Let D_∞ denote the infinite dihedral group, defined by the presentation

$$D_\infty = \langle x, y \mid x^2 = y^2 = 1 \rangle.$$

Lemma 6.1. *There is an epimorphism $\phi: \mathcal{M}(N_4) \rightarrow D_\infty$.*

Proof. According to the main result of [21] simplified in [18], $\mathcal{M}(N_4)$ admits a presentation with generators t_{ε_2} , t_{δ_i} , u_i for $i = 1, 2, 3$ and relations (R1, R3, R4, R6, R7, R9, R10, R11, R12) and

$$\begin{aligned} t_{\delta_{i+1}} u_1 u_{i+1} &= u_i u_{i+1} t_{\delta_i} \quad \text{for } i = 1, 2 \\ (t_{\varepsilon_2} u_3)^2 &= 1, \quad t_{\delta_1} (t_{\delta_2} t_{\delta_3} u_3 u_2) t_{\delta_1} = (t_{\delta_2} t_{\delta_3} u_3 u_2). \end{aligned}$$

It is easy to check that the mapping $\phi(t_{\varepsilon_2}) = xy$, $\phi(t_{\delta_i}) = 1$, $\phi(u_i) = y$ for $i = 1, 2, 3$, respects the defining relations of $\mathcal{M}(N_4)$, hence it defines a homomorphism onto D_∞ . \square

Corollary 6.2. *For $h \geq 3$ there is a homomorphism $f: \mathcal{M}(N_4) \rightarrow \mathcal{M}(N_h)$, such that $\mathrm{Im}(f)$ is isomorphic to D_∞ .*

Proof. Fix $h \geq 3$. By the proof of [20, Theorem 3], t_{δ_1} can be written in $\mathcal{M}(N_h)$ as a product of two involutions σ, τ . Since t_{δ_1} has infinite order in $\mathcal{M}(N_h)$, the mapping $x \mapsto \sigma$, $y \mapsto \tau$ defines an embedding $D_\infty \rightarrow \mathcal{M}(N_h)$. By pre-composing this embedding with the epimorphism ϕ from Lemma 6.1, we obtain f . \square

The following two theorems can be proved by the same method as Theorem 1.4. We leave the details to the reader.

Theorem 6.3. *Suppose that $g \geq 5$, $g \geq 2h + 2$ and $f: \mathcal{M}(N_g) \rightarrow \mathcal{M}(N_h)$ is a homomorphism. Then $\mathrm{Im}(f)$ is abelian.*

Theorem 6.4. *Suppose that $g \geq 3$ and $h \leq 2g$. Then the only homomorphism from $\mathcal{M}(S_g)$ to $\mathcal{M}(N_h)$ is the trivial one.*

7. HOMOMORPHISMS FROM $\mathcal{M}(N_g)$ TO $\mathrm{GL}(g-1, \mathbb{C})$.

The aim of this section is to prove Theorem 1.5. The prove it divided in two cases, according to the parity of the genus.

Let $g = 2r + s$, $s \in \{1, 2\}$, $S' = S_{r,s}$ and $\iota: \mathcal{M}(S') \rightarrow \mathcal{M}(N_{g,n})$ be the homomorphism from Corollary 3.6. If $f: \mathcal{M}(N_{g,n}) \rightarrow \mathrm{GL}(m, \mathbb{C})$ is a homomorphism, then we set $f' = f \circ \iota$.

Proof of Theorem 1.5 for odd g . Suppose that $N = N_{2r+1}$, $r \geq 3$ and $f: \mathcal{M}(N) \rightarrow \mathrm{GL}(2r, \mathbb{C})$ is a homomorphism, such that $\mathrm{Im}(f)$ is not abelian. By Theorem 1.2, f' is conjugate to the homological representation Φ , and thus there exists a basis, such that $f(t_{\varepsilon_i}) = f'(t_{\alpha_i}) = A_i$, $f(t_{\delta_{2i}}) = f'(t_{\beta_i}) = B_i$ for $1 \leq i \leq r$ and $f(t_{\delta_{2j+1}}) = f'(t_{\gamma_j}) = C_j$ for $1 \leq j \leq r-1$. Set $U_k = f(u_k)$ for $1 \leq k \leq 2r$.

Since U_{2r} commutes with A_i and B_i for $1 \leq i \leq r$, and with C_j for $j = 1, \dots, r-2$ (R6, R8) thus, by Lemma 2.1,

$$U_{2r} = \begin{pmatrix} \lambda I_{2r-2} & 0 \\ 0 & X \end{pmatrix},$$

for some 2×2 matrix X . Since U_{2r} is conjugate to U_{2r}^{-1} we have $\lambda \in \{-1, 1\}$ and by multiplying f by $(-1)^{\mathrm{ab}}$ if necessary, we may assume $\lambda = 1$. The relation $B_r U_{2r} B_r = U_{2r}$ (R12) implies $X = \begin{pmatrix} x & 0 \\ y & -x \end{pmatrix}$.

From (R11) and (R7) we have

$$\begin{aligned} U_{2r-2} &= (C_{r-1} B_r B_{r-1} C_{r-1})^{-1} U_{2r} (C_{r-1} B_r B_{r-1} C_{r-1}), \\ U_{2r} U_{2r-2} - U_{2r-2} U_{2r} &= 0, \end{aligned}$$

and since the left hand side of the last equation is equal to

$$(1 - x^2)(E_{2r, 2r-3} + E_{2r-2, 2r-1}),$$

thus $x^2 = 1$. We have $U_{2r}^{-1} = U_{2r}$, and from (R11) and (R9)

$$\begin{aligned} U_{2r-1} &= (C_{r-1} B_r)^{-1} U_{2r} (C_{r-1} B_r), \\ U_{2r} U_{2r-1} U_{2r} - U_{2r-1} U_{2r} U_{2r-1} &= 0. \end{aligned}$$

By considering the cases $x = 1$ and $x = -1$ separately, we find that the left hand side of the last equation is of the form $(y - x)^2 Z$, where $Z \neq 0$. Hence $x = y$ and $U_{2r} = \Psi_1(u_{2r})$ if $x = 1$, or $U_{2r} = \Psi_2(u_{2r})$ if $x = -1$. By Theorem 3.1, f is equal to Ψ_1 or Ψ_2 on generators of $\mathcal{M}(N)$. \square

Now we will borrow some arguments from [15] to prove Lemma 7.3 below, which will be a starting point for the proof of Theorem 1.5 for even genus.

Lemma 7.1. *Suppose that $n \leq 1$, $g \geq 5$ and $f: \mathcal{M}(N_{g,n}) \rightarrow \mathrm{GL}(m, \mathbb{C})$ is a homomorphism. If there is a flag $0 = W_0 \subset W_1 \subset \dots \subset W_k = \mathbb{C}^m$ of $\mathcal{M}(N_{g,n})$ -invariant subspaces such that $\dim(W_i/W_{i-1}) < g - 1$ for $i = 1, \dots, k$, then $\mathrm{Im}(f)$ is abelian.*

Proof. The same argument as in the proof of [15, Lemma 4.8] can be applied, using Theorem 1.3, to show that with respect to some basis $f[\mathcal{M}(N_{g,n}), \mathcal{M}(N_{g,n})]$ is contained in the subgroup of upper triangular matrices with 1 on the diagonal. Since this subgroup is nilpotent and $[\mathcal{M}(S'), \mathcal{M}(S')]$ is perfect, it follows that $f'[\mathcal{M}(S'), \mathcal{M}(S')]$ is trivial, which means that $\mathrm{Im}(f')$ is abelian, and so is $\mathrm{Im}(f)$. \square

Lemma 7.2. *Suppose that $N = N_{2r+2}$, $r \geq 3$ and $f: \mathcal{M}(N) \rightarrow \mathrm{GL}(2r+1, \mathbb{C})$ is a homomorphism, such that $\mathrm{Im}(f)$ is not abelian. Then $L_1 = f(t_{\delta_1})$ has an eigenvalue λ such that $\dim E(L_1, \lambda) = 2r$.*

Proof. By [15, Corollary 4.6] applied to f' , L_1 has at most two eigenvalues. It follows that there is an eigenvalue λ with $\#\lambda \geq r+1 \geq 4$. Set $m = \dim E(L_1, \lambda)$. Since $\mathrm{Im}(f)$ is not abelian, thus $m \leq 2r$. We are going to show that $m = 2r$.

Let R be the subsurface obtained by removing from N a regular neighbourhood of $\delta_1 \cup \delta_2$. We have $R \approx N_{2r,1}$. We treat $\mathcal{M}(R)$ as a subgroup of $\mathcal{M}(N)$.

Suppose $m \leq 2r-2$. Let $W = E^k(L_1, \lambda)$, where $k = \max\{4-m, 1\}$. Observe that W is a $\mathcal{M}(R)$ -invariant subspace with $3 \leq \dim W \leq 2r-2$. By Lemma 7.1, $f(\mathcal{M}(R))$ is abelian, which means $f(t_{\delta_4}) = f(t_{\delta_5})$. By Lemma 3.3, $\mathrm{Im}(f)$ is abelian, a contradiction.

Suppose that $m = 2r-1$ and set $L_2 = f(t_{\delta_2})$. If $E(L_1, \lambda) \neq E(L_2, \lambda)$ then $E(L_1, \lambda) \cap E(L_2, \lambda)$ is a $\mathcal{M}(R)$ -invariant subspace of dimension $2r-3$ or $2r-2$ and we can use the same argument as above to obtain a contradiction. If $E(L_1, \lambda) = E(L_2, \lambda)$, then by [15, Lemma 4.3] applied to f' , $E(L_1, \lambda)$ is a $\mathcal{M}(S')$ -invariant subspace of dimension $2r-1$, and by [15, Lemma 4.8] f' is trivial. It follows that $\mathrm{Im}f$ is abelian, a contradiction. \square

Lemma 7.3. *Suppose that $N = N_{2r+2}$, $r \geq 3$ and $f: \mathcal{M}(N) \rightarrow \mathrm{GL}(2r+1, \mathbb{C})$ is a homomorphism. If $r = 3$ then assume that 1 is the unique eigenvalue of $f(t_{\delta_1})$. Then either $\mathrm{Im}(f)$ is abelian, or with respect to some basis $f(t_{\varepsilon_i}) = A_i$, $f(t_{\delta_{2i}}) = B_i$ for $i = 1, \dots, r$.*

Proof. Suppose that $\mathrm{Im}(f)$ is not abelian. By Lemma 7.2, $L_1 = f(t_{\delta_1})$ has an eigenvalue λ with $\dim E(L_1, \lambda) = 2r$. If $r = 3$ then $\lambda = 1$

by assumption, and for $r \geq 4$, $\lambda = 1$ by the proof of [15, Lemma 5.2]. Since $\mathcal{M}(S')$ is perfect, thus $\det L_1 = 1$ and $\lambda = 1$ is the unique eigenvalue. Set $L_2 = f(t_{\delta_2})$. We claim that $E(L_1, 1) \neq E(L_2, 1)$. For otherwise it is easy to prove that L_1 and L_2 commute (see the proof of Theorem 1.3 for $(g, m) = (6, 4)$, case (5)), and $\text{Im}(f)$ is abelian by Lemma 3.3, a contradiction. Now we can apply [15, Lemma 4.7] to f' to conclude that with respect to some basis we have $f(t_{\varepsilon_i}) = f'(t_{\alpha_i}) = A_i$, $f(t_{\delta_{2i}}) = f'(t_{\beta_i}) = B_i$ for $i = 1, \dots, r$. \square

Proof of Theorem 1.5 for even g . Suppose that $N = N_{2r+2}$, $r \geq 4$ and $f: \mathcal{M}(N) \rightarrow \text{GL}(2r+1, \mathbb{C})$ is a homomorphism, such that $\text{Im}(f)$ is not abelian. By Lemma 7.3 there is a basis such that $f(t_{\varepsilon_i}) = A_i$ and $f(t_{\delta_{2i}}) = B_i$ for $1 \leq i \leq r$. Set $D_i = f(t_{\delta_{2i+1}})$ for $1 \leq i \leq r$ and $U_j = f(u_j)$ for $1 \leq j \leq 2r+1$.

Fix $i \in \{1, \dots, r-1\}$. Since D_i is conjugate to A_1 , it has one eigenvalue $\lambda = 1$. For $j \notin \{i, i+1\}$ the relations $D_i A_j = A_j D_i$ and $D_i B_j = B_j D_i$ imply, by Lemma 2.1, that D_i has the form

$$D_i = \begin{pmatrix} I_{2(i-1)} & 0 & 0 & 0 & 0 \\ 0 & F_{11} & F_{12} & 0 & X_1 \\ 0 & F_{21} & F_{22} & 0 & X_2 \\ 0 & 0 & 0 & I_{2(g-i-1)} & 0 \\ 0 & Y_1 & Y_2 & 0 & z \end{pmatrix},$$

where F_{kl} are 2×2 matrices, X_k are 2×1 vectors, Y_l are 1×2 vectors and z is a complex number. The relations $D_i A_i = A_i D_i$ and $D_i A_{i+1} = A_{i+1} D_i$ imply, for $k, l \in \{1, 2\}$, $V F_{kl} = F_{kl} V$, $V F_{kl} = F_{kl}$ for $k \neq l$, $V X_k = X_k$, $Y_l V = Y_l$, hence

$$\begin{aligned} F_{11} &= \begin{pmatrix} s_1 & t_1 \\ 0 & s_1 \end{pmatrix}, F_{12} = \begin{pmatrix} 0 & v_1 \\ 0 & 0 \end{pmatrix}, X_1 = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}, \\ F_{21} &= \begin{pmatrix} 0 & v_2 \\ 0 & 0 \end{pmatrix}, F_{22} = \begin{pmatrix} s_2 & t_2 \\ 0 & s_2 \end{pmatrix}, X_2 = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}, \\ Y_1 &= (0 \quad y_1), Y_2 = (0 \quad y_2). \end{aligned}$$

Since s_1, s_2 are eigenvalues, we have $s_1 = s_2 = 1$ and $\det D_i = z$, which gives $z = 1$. Now, by solving the equations $B_i D_i B_i - D_i B_i D_i = 0$ and $B_{i+1} D_i B_{i+1} - D_i B_{i+1} D_i = 0$ we obtain $t_1 = t_2 = 1$, $v_1 v_2 = 1$, $y_2 = y_1 v_1$,

$x_2 = x_1 v_2$, $x_1 y_1 = 0$. Thus, for $i = 1, \dots, r-1$ we have

$$D_i = \begin{pmatrix} I_{2(i-1)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & \alpha_i & 0 & \alpha_i x_i \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_i^{-1} & 1 & 1 & 0 & x_i \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{2(g-i-1)} & 0 \\ 0 & 0 & y_i & 0 & \alpha_i y_i & 0 & 1 \end{pmatrix}, \quad x_i y_i = 0.$$

Similarly, using the relations between D_r and A_i , B_i it can be shown that

$$D_r = \begin{pmatrix} I_{2g-2} & 0 & 0 & 0 \\ 0 & 1 & 1 & x_r \\ 0 & 0 & 1 & 0 \\ 0 & 0 & y_r & 1 \end{pmatrix}, \quad x_r y_r = 0.$$

It is not possible that $x_r = y_r = 0$, because then $D_r = A_r$ and Lemma 3.4 would give a contradiction. For $1 \leq i \leq r-1$, by solving the equation $D_i D_r - D_r D_i = 0$ we obtain $x_i y_r = 0$ and $x_r y_i = 0$. It follows that either $x_i = 0$ for all $i = 1, \dots, r$, or $y_i = 0$ for all $i = 1, \dots, r$. We are going to show that it is possible to change the basis so that $\alpha_i = -1$ for $i = 1, \dots, r-1$ and $x_r + y_r = -2$. Suppose that the old basis is $\beta_1 = (v_1, w_1, \dots, v_r, w_r, v_{r+1})$. We consider two cases.

Case 1: $x_r = 0$. Then $y_r \neq 0$ and the new basis is:

$$\begin{aligned} v'_i &= (-1)^{r-i} \alpha_i \cdots \alpha_{r-1} v_i, \quad w'_i = (-1)^{r-i} \alpha_i \cdots \alpha_{r-1} w_i, \quad i = 1, \dots, r-1, \\ v'_r &= v_r, \quad w'_r = w_r, \quad v'_{r+1} = -\frac{y_r}{2} v_{r+1}. \end{aligned}$$

In the new basis we have:

$$D_r = \Psi_1(t_{\delta_{2r+1}}), \quad D_i = C_i + x'_i (E_{2r+1, 2i} - E_{2r+1, 2i+2}),$$

for $i = 1, \dots, r-1$.

Case 2: $y_r = 0$. Then $x_r \neq 0$ and the new basis is:

$$\begin{aligned} v'_i &= (-1)^{r-i+1} \alpha_i \cdots \alpha_{r-1} \frac{x_r}{2} v_i, \quad w'_i = (-1)^{r-i+1} \alpha_i \cdots \alpha_{r-1} \frac{x_r}{2} w_i, \\ i &= 1, \dots, r-1, \quad v'_r = -\frac{x_r}{2} v_r, \quad w'_r = -\frac{x_r}{2} w_r, \quad v'_{r+1} = v_{r+1}. \end{aligned}$$

In the new basis we have:

$$D_r = \Psi_2(t_{\delta_{2r+1}}), \quad D_i = C_i + x'_i (E_{2i-1, 2r+1} - E_{2i+1, 2r+1}),$$

for $i = 1, \dots, r-1$.

Since U_{2r+1} commutes with A_i and B_i for $1 \leq i \leq r-1$, thus, by Lemma 2.1,

$$U_{2r+1} = \text{diag}(\lambda_1 I_2, \lambda_2 I_2, \dots, \lambda_{r-1} I_2, X),$$

for some 3×3 matrix X . The relations $A_r U_{2r+1} = U_{2r+1} A_r$ (R8) and $D_r U_{2r+1} D_r = U_{2r+1}$ (R12) imply that X has the form

$$X = \begin{pmatrix} \lambda_r & \alpha & \lambda_r \\ 0 & \lambda_r & 0 \\ 0 & \beta & -\lambda_r \end{pmatrix} \quad \text{or} \quad X = \begin{pmatrix} \lambda_r & \alpha & \beta \\ 0 & \lambda_r & 0 \\ 0 & \lambda_r & -\lambda_r \end{pmatrix}$$

respectively in case 1 and case 2. For $1 \leq i \leq r-1$, by the relation (R6) we have $D_i U_{2r+1} - U_{2r+1} D_i = 0$. By solving this equation we obtain $\lambda_i = \lambda_{i+1}$ and $x'_i = 0$, hence $D_i = C_i$. We also see that U_{2r+1} has two eigenvalues $\lambda_r, -\lambda_r$ with $\# \lambda_r = 2r$. Since U_{2r+1} is conjugate to U_{2r+1}^{-1} we have $\lambda_r \in \{-1, 1\}$ and by multiplying f by $(-1)^{\text{ab}}$ if necessary, we may assume $\lambda_r = 1$.

By the relation (R11) we have

$$\begin{aligned} U_{2r} &= (B_r C_r)^{-1} U_{2r+1}^{-1} (B_r C_r), \\ U_{2r-1} &= (B_r C_r C_{r-1} B_r)^{-1} U_{2r+1} (B_r C_r C_{r-1} B_r). \end{aligned}$$

Similarly as in the proof for odd g , by solving $U_{2r+1} U_{2r-1} - U_{2r-1} U_{2r+1} = 0$ we obtain $\beta = -2\alpha$, and then by solving $U_{2r+1} U_{2r} U_{2r+1} - U_{2r} U_{2r+1} U_{2r} = 0$ we obtain $\alpha = -1$ in the case 1, or $\alpha = 1$ in the case 2. Hence $U_{2r+1} = \Psi_1(u_{2r+1})$ in the case 1, or $U_{2r+1} = \Psi_2(u_{2r+1})$ in the case 2. By Theorem 3.1, f is equal to Ψ_1 in the case 1, and equal to Ψ_2 in the case 2, on generators of $\mathcal{M}(N)$. \square

8. HOMOMORPHISMS FROM $\mathcal{M}(N_8)$ TO $\text{GL}(7, \mathbb{C})$

The aim of this section is to prove Theorem 1.6. First we have to define the epimorphism $\epsilon: \mathcal{M}(N_{2r+2}) \rightarrow \text{Sp}(2r, \mathbb{Z}_2)$.

Fix $r \geq 1$ and set $V = H_1(N_{2r+2}, \mathbb{Z}_2)$. V is a vector space over \mathbb{Z}_2 of dimension $2r + 2$ with basis $\overline{x}_i = [\xi_i]_2$ for $1 \leq i \leq 2r + 2$, where $[\xi_i]_2$ denotes the mod 2 homology class of the curve ξ_i . The mod 2 intersection pairing is the symmetric bilinear form on V satisfying $\langle \overline{x}_i, \overline{x}_j \rangle_2 = \delta_{ij}$. We define another basis for V . For $1 \leq i \leq r$ we set

$$\begin{aligned} v_i &= [\varepsilon_i]_2 = \overline{x}_1 + \cdots + \overline{x}_{2i}, & w_i &= [\delta_{2i}]_2 = \overline{x}_{2i} + \overline{x}_{2i+1}, \\ c &= \overline{x}_{2r+2}, & d &= \overline{x}_1 + \cdots + \overline{x}_{2r+2}. \end{aligned}$$

Let $\text{Iso}(V)$ denote the group of automorphisms of V preserving $\langle \cdot, \cdot \rangle_2$.

Lemma 8.1. *The group $\text{Iso}(V)$ is isomorphic to a semi-direct product $\text{Sp}(2r, \mathbb{Z}_2) \ltimes \mathbb{Z}_2^{2r+1}$.*

Proof. It is easy to check that d is the unique vector of V satisfying $\langle x, d \rangle_2 = \langle x, x \rangle_2$ for all $x \in V$, which implies that d is fixed by all elements of $\text{Iso}(V)$.

Let $W = \text{span}\{v_i, w_i \mid i = 1, \dots, r\}$ and observe that the restriction of $\langle \cdot, \cdot \rangle_2$ to W is nondegenerate and $\langle x, x \rangle_2 = 0$ for $x \in W$, hence it is a symplectic form on W . For $R \in \text{Sp}(W)$ we define $A_R \in \text{Iso}(V)$ as

$$A_R(d) = d, \quad A_R(c) = c, \quad A_R(x) = R(x) \quad \text{for } x \in W.$$

It is easy to check that $W = \{x \in V \mid \langle x, d \rangle_2 = \langle x, c \rangle_2 = 0\}$. It follows that if $L \in \text{Iso}(V)$ fixes c , then since $L(d) = d$, L preserves W , and hence $L = A_R$ for some $R \in \text{Sp}(W)$. Thus the mapping $R \mapsto A_R$ defines an isomorphism $\text{Sp}(W) \rightarrow \text{Stab}_{\text{Iso}(V)}(c)$.

For $x \in \mathbb{Z}_2$ and $z \in W$ we define $B_{x,z} \in \text{Iso}(V)$ as

$$B_{x,z}(d) = d, \quad B_{x,z}(c) = c + xd + z, \quad B_{x,z}(w) = w + \langle w, z \rangle_2 d \quad \text{for } w \in W.$$

Let

$$N = \{B_{x,z} \mid x \in \mathbb{Z}_2, z \in W\}.$$

This is a subgroup of $\text{Iso}(V)$ with the group law

$$B_{x_1, z_1} B_{x_2, z_2} = B_{x_1 + x_2 + \langle z_1, z_2 \rangle_2, z_1 + z_2}.$$

It follows that N is abelian and $B_{x,z}^2 = 1$ for all x, z . Thus N is isomorphic to \mathbb{Z}_2^{2r+1} .

Let $L \in \text{Iso}(V)$ be arbitrary. Since $\langle L(c), d \rangle = \langle L(c), L(d) \rangle = \langle c, d \rangle = 1$, thus $L(c) = c + xd + z$ for some $x \in \mathbb{Z}_2, z \in W$. It follows that $B_{x,z}^{-1}L \in \text{Stab}_{\text{Iso}(V)}(c)$ and hence $L = B_{x,z}A_R$ for some $R \in \text{Sp}(W)$. This decomposition is clearly unique, and since $A_R B_{x,z} A_R^{-1} = B_{x, R(z)}$, thus N is normal in $\text{Iso}(V)$ and $\text{Iso}(V) = N \rtimes \text{Stab}_{\text{Iso}(V)}(c)$. \square

Lemma 8.2. *For $r \geq 2$ there is an epimorphism*

$$\epsilon: \mathcal{M}(N_{2r+2}) \rightarrow \text{Sp}(2r, \mathbb{Z}_2),$$

whose kernel is normally generated by $t_{\delta_{2r+1}} u_{2r+1}$ and $t_{\delta_{2r+1}} t_{\epsilon_r}^{-1}$.

Proof. Let $\mathcal{M} = \mathcal{M}(N_{2r+2})$. The action of \mathcal{M} on $V = H_1(N_{2r+2}, \mathbb{Z}_2)$ induces a homomorphism $\rho: \mathcal{M} \rightarrow \text{Iso}(V)$, which was proved to be surjective in [8] and [17], and whose kernel is the normal closure of $t_{\delta_{2r+1}} u_{2r+1}$ by [23]. By Lemma 8.1, there exists a normal subgroup N of $\text{Iso}(V)$, such that $\text{Iso}(V)/N$ is isomorphic to $\text{Sp}(2r, \mathbb{Z}_2)$. We define ϵ to be the composition of ρ with the canonical projection $\text{Iso}(V) \rightarrow \text{Iso}(V)/N$.

Let K be the normal closure of $t_{\delta_{2r+1}} u_{2r+1}$ and $t_{\delta_{2r+1}} t_{\epsilon_r}^{-1}$ in \mathcal{M} . We claim that $K \subseteq \ker \epsilon$. We have $t_{\delta_{2r+1}} u_{2r+1} \in \ker \rho \subset \ker \epsilon$. For $x \in V$ we have $\rho(t_{\epsilon_r})(x) = x + \langle v_r, x \rangle_2 v_r$ and $\rho(t_{\delta_{2r+1}})(x) = x + \langle [\delta_{2r+1}]_2, x \rangle [\delta_{2r+1}]_2$. Since $[\delta_{2r+1}]_2 = v_r + d$, it is not difficult to check

that $\rho(t_{\delta_{2r+1}}) = B_{1,v_r} \circ \rho(t_{\varepsilon_r})$, which gives $\rho(t_{\delta_{2r+1}} t_{\varepsilon_r}^{-1}) \in N$ and $t_{\delta_{2r+1}} t_{\varepsilon_r}^{-1} \in \ker \epsilon$. It follows that there is an induced epimorphism

$$\epsilon': \mathcal{M}/K \rightarrow \text{Iso}(V)/N \cong \text{Sp}(2r, \mathbb{Z}_2).$$

To prove that ϵ' is an isomorphism, it suffices to show $[\mathcal{M} : K] \leq |\text{Sp}(2r, \mathbb{Z}_2)|$. We are going to prove the last inequality by exhibiting an epimorphism $\text{Sp}(2r, \mathbb{Z}_2) \rightarrow \mathcal{M}/K$.

Observe that the map $\eta: \mathcal{M}(S') \rightarrow \mathcal{M}/K$ defined to be the composition of $\iota: \mathcal{M}(S') \rightarrow \mathcal{M}$ from Corollary 3.6 with the canonical projection $\pi: \mathcal{M} \rightarrow \mathcal{M}/K$ is surjective, because \mathcal{M} is generated by twists about curves on $P(S')$ and $t_{\delta_{2r+1}} u_{2r+1}$ by Theorem 3.1. Gluing a disc along the boundary component of S' bounding a pair of pants with α_r and γ_r induces an epimorphism $\mathcal{M}(S') \rightarrow \mathcal{M}(S_{r,1})$ whose kernel is normally generated by $t_{\gamma_r} t_{\alpha_r}^{-1}$ (see [15, Proposition 3.8]). Since $\iota(t_{\gamma_r} t_{\alpha_r}^{-1}) = t_{\delta_{2r+1}} t_{\varepsilon_r}^{-1} \in K$, it follows that we have an induced epimorphism $\eta': \mathcal{M}(S_{r,1}) \rightarrow \mathcal{M}/K$. There is an epimorphism $\mathcal{M}(S_{r,1}) \rightarrow \text{Sp}(2r, \mathbb{Z}_2)$ induced by the action of $\mathcal{M}(S_{r,1})$ on $H_1(S_{r,1}, \mathbb{Z}_2)$, whose kernel is normally generated by $t_{\alpha_1}^2$ (see [2, Theorem 5.7], here we are using the assumption $r \geq 2$). By applying Lemma 3.4 (with $i = r, j = 2r + 1$) to $\pi: \mathcal{M} \rightarrow \mathcal{M}/K$, we have $\eta'(t_{\alpha_1}^2) = \pi(t_{\delta_1}^2) = 1$. It follows that there is an induced epimorphism $\eta'': \text{Sp}(2r, \mathbb{Z}_2) \rightarrow \mathcal{M}/K$.

$$\begin{array}{ccccc} \mathcal{M}(S') & \xrightarrow{\iota} & \mathcal{M} & \xrightarrow{\pi} & \mathcal{M}/K \\ \downarrow & & \nearrow \eta' & & \\ \mathcal{M}(S_{r,1}) & & \nearrow \eta'' & & \\ \downarrow & & & & \\ \text{Sp}(2r, \mathbb{Z}_2) & & & & \end{array}$$

The existence of η'' proves that ϵ' is an isomorphism and $K = \ker \epsilon$. \square

Lemma 8.3. *Suppose that $f: \mathcal{M}(N_8) \rightarrow \text{GL}(7, \mathbb{C})$ is a homomorphism, such that $f(t_{\delta_1})$ has order 2. Then f or $(-1)^{\text{ab}} f$ factors through the epimorphism $\epsilon: \mathcal{M}(N_8) \rightarrow \text{Sp}(6, \mathbb{Z}_2)$.*

Proof. Let H be the normal closure of $t_{\delta_1}^2$ in $\mathcal{M} = \mathcal{M}(N_8)$ and $G = \mathcal{M}/H$. Since $H \subseteq \ker f$, we have a homomorphism $f': G \rightarrow \text{GL}(7, \mathbb{C})$ such that $f = f' \circ \pi$, where $\pi: \mathcal{M} \rightarrow G$ is the canonical projection. There is a homomorphism $\rho: \mathfrak{S}_8 \rightarrow G$, defined as $\rho(\sigma_i) = \pi(t_{\delta_i})$, where $\sigma_i = (i, i+1)$, for $1 \leq i \leq 7$. Let $\phi: \mathfrak{S}_8 \rightarrow \text{GL}(7, \mathbb{C})$ be the composition $\phi = f' \circ \rho$. If ϕ is reducible, then $\text{Im}(\phi)$ is abelian by Lemma 2.2, $f(t_{\delta_1}) = \phi(\sigma_1) = \phi(\sigma_2) = f(t_{\delta_2})$, and $\text{Im}(f)$ is also abelian by Lemma 3.3, which implies $f(t_{\delta_1}) = 1$ by Theorem 3.2, a contradiction. Hence

ϕ is irreducible and since $\det f(t_{\delta_1}) = 1$ (by Theorem 3.2), ϕ is the tensor product of the standard and sign representations (by Lemma 2.2). For $1 \leq i \leq 7$ set $L_i = f(t_{\delta_i}) = \phi(\sigma_i)$. With respect to some basis (v_1, \dots, v_7) we have

$$L_1 = \text{diag}(A, -I_5), \quad L_7 = \text{diag}(-I_5, B), \quad L_i = \text{diag}(-I_{i-2}, C, -I_{6-i})$$

for $2 \leq i \leq 6$, where

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let M be the matrix of $f(\varepsilon_3)$. Since M commutes with L_i for $i \neq 6$ (R5), it preserves $E(L_i, 1) = \text{span}\{v_i\}$. Hence $M(v_i) = x_i v_i$ for $i \neq 6$ and $M(v_6) = y_1 v_1 + \dots + y_7 v_7$, for some complex numbers x_i, y_j . By solving the equations $ML_i = L_i M$ for $1 \leq i \leq 5$ and $i = 7$ we obtain

$$x_i = x_1, \quad y_i = i y_1 \quad \text{for } 1 \leq i \leq 5, \quad y_6 = x_1 + 6y_1, \quad x_7 = y_6 - 2y_7.$$

Since M and L_i are conjugate, they have the same eigenvalues, which gives $x_1 = -1$ and $y_6 = -x_7$. If $y_6 = 1$, then $y_1 = 1/3$, $y_7 = 1$, which contradicts the braid relation $ML_6 M = L_6 M L_6$ (R5). Hence $y_6 = -1$, $y_1 = 0$, $y_7 = -1$, which means $M = L_7$.

For $i = 1, \dots, 7$ let U_i be the matrix of $f(u_i)$. Since U_7 commutes with L_j for $1 \leq j \leq 5$ (R6) and with $M = L_7$ (R8), we obtain, as above, that

$$\begin{aligned} U_7(v_i) &= x v_i \quad \text{for } 1 \leq i \leq 5, \\ U_7(v_6) &= y(v_1 + 2v_2 + 3v_3 + 4v_4 + 5v_5) + (x + 6y)v_6 + z v_7 \\ U_7(v_7) &= (x + 6y - 2z)v_7 \end{aligned}$$

for some complex numbers x, y, z . Since U_7 is conjugate to its inverse, and x is an eigenvalue of multiplicity at least 5, thus $x = \pm 1$, and by multiplying f by $(-1)^{\text{ab}}$ if necessary, we may assume $x = -1$. By (R11) we have $U_5 = (L_6 L_7 L_5 L_6)^{-1} U_7 (L_6 L_7 L_5 L_6)$ and by solving $U_5 U_7 = U_7 U_5$ we obtain $y = 0$. Since $\det U_7 = \pm 1$, either $-1 - 2z = 1$ or $-1 - 2z = -1$. In the latter case we have $U_7 = -I$, and since U_6 is conjugate to U_7 , thus $U_6 = -I$, and the relation $L_6 U_7 U_6 = U_7 U_6 L_7$ (R10) gives $L_6 = L_7$, a contradiction. Hence $z = -1$ and $U_7 = L_7$.

We have $M = U_7 = L_7$ and since $L_7^2 = I$, thus $\{t_{\delta_7} t_{\varepsilon_3}^{-1}, t_{\delta_7} u_7\} \subset \ker f$, which implies, by Lemma 8.2, that f factors through ϵ . \square

Proof of Theorem 1.6. Suppose that $f: \mathcal{M}(N_8) \rightarrow \text{GL}(7, \mathbb{C})$ is a homomorphism, such that $\text{Im}(f)$ is not abelian. By Lemma 7.2, $L = f(t_{\delta_1})$ has an eigenvalue λ such that $\dim E(L, \lambda) = 6$. Since L is conjugate to L^{-1} we have $\lambda^2 = 1$. Suppose that $\lambda = -1$. Then since $\det L = 1$

we have $\#\lambda = 6$, and there is another eigenvalue $\mu = 1$. It follows that L has order 2 and the case (2) holds by Lemma 8.3. If $\lambda = 1$ then it must be the unique eigenvalue, and the case (3) holds by Lemma 7.3 and the proof of Theorem 1.5 for even g . \square

Remark 8.4. Suppose that G is a finite quotient of $\mathcal{M}(N_g)$ for $g \geq 7$, $g \neq 8$, and $f: G \rightarrow \mathrm{GL}(g-1, \mathbb{C})$ is a homomorphism. Then, by Theorem 1.5, $\mathrm{Im}(f)$ is abelian, and if G is perfect, then f must be trivial. For example, by Lemma 8.2, for $r \geq 4$, the only homomorphism from $\mathrm{Sp}(2r, \mathbb{Z}_2)$ to $\mathrm{GL}(2r+1, \mathbb{C})$ is the trivial one.

REFERENCES

- [1] J. Aramayona, J. Souto. Homomorphisms between mapping class groups. *Geom. Topol.* 16 (2012), 2285–2341.
- [2] J. A. Berrick, V. Gebhardt, L. Paris. Finite index subgroups of mapping class groups. To appear in *Proc. London Math. Soc.*
- [3] J. S. Birman, D. R. J. Chillingworth. On the homeotopy group of a non-orientable surface. *Proc. Camb. Philos. Soc.* 71 (1972), 437–448.
- [4] D. R. J. Chillingworth. A finite set of generators for the homeotopy group of a non-orientable surface. *Proc. Camb. Phil. Soc.* 65 (1969), 409–430.
- [5] D. B. A. Epstein. Curves on 2-manifolds and isotopies. *Acta Math.* 115 (1966), 83–107.
- [6] J. Franks, M. Handel. Triviality of some representations of $\mathrm{MCG}(S)$ in $\mathrm{GL}(n, \mathbb{C})$, $\mathrm{Diff}(S^2)$ and $\mathrm{Homeo}(T^2)$. To appear in *Proc. AMS*.
- [7] W. Fulton, J. Harris. *Representation theory: A first course*. Springer-Verlag 1991.
- [8] S. Gadgil and D. Pancholi. Homeomorphisms and the homology of non-orientable surfaces. *Proc. Indian Acad. Sci. Math. Sci.* **115** (2005), 251257.
- [9] P. A. Gastesi. A note on Torelli spaces of compact non-orientable Klein surfaces. *Ann. Acad. Sci. Fenn. Math.* 24 (1999) 23–30.
- [10] W. Harvey, M. Korkmaz. Homomorphisms from mapping class groups. *Bull. London Math. Soc.* 37 (2005), 275–284.
- [11] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge 2002.
- [12] M. Korkmaz. First homology group of mapping class group of nonorientable surfaces. *Math. Proc. Camb. Phil. Soc.* 123 (1998), 487–499.
- [13] M. Korkmaz. Problems on homomorphisms of mapping class groups. In: *Problems on Mapping Class Groups and Related Topics*, B. Farb Ed., *Proc. Symp. Pure Math.* 74 (2006), 85–94.
- [14] M. Korkmaz, Low-dimensional linear representations of mapping class groups. *arXiv:1104.4816*
- [15] M. Korkmaz, The symplectic representation of the mapping class group is unique. *arXiv:1108.3241*
- [16] W. B. R. Lickorish. Homeomorphisms of non-orientable two-manifolds. *Proc. Camb. Phil. Soc.* 59 (1963), 307–317.
- [17] J. D. McCarthy, U. Pinkall, Representing homology automorphisms of nonorientable surfaces, Max Planck Inst. preprint MPI/SFB 85-11, revised version written in 2004. Available at <http://www.math.msu.edu/~mccarthy>.

- [18] L. Paris, B. Szepietowski. A presentation for the mapping class group of a nonorientable surface. In preparation.
- [19] M. Stukow. Generating mapping class groups of nonorientable surfaces with boundary. *Adv. Geom.* 10 (2010), 249–273.
- [20] B. Szepietowski. Mapping class group of a non-orientable surface and moduli space of Klein surfaces. *C. R. Acad. Sci. Paris, Ser. I* 335 (2002), 1053–1056.
- [21] B. Szepietowski. A presentation for the mapping class group of the closed non-orientable surface of genus 4. *J. Pure Appl. Algebra* 213 (2009), 2001–2016.
- [22] B. Szepietowski. Embedding the braid group in mapping class groups. *Publ. Mat.* 54 (2010), 359–368.
- [23] B. Szepietowski. Crosscap slides and the level 2 mapping class group of a nonorientable surface. *Geom. Dedicata* 160 (2012) 169–183.
- [24] R. A. Wilson, P. Walsh, J. Tripp, I. Suleiman, S. Rogers, R. Parker, S. Norton, S. Linton, J. Bray. Atlas of finite group representations. Online database available at <http://brauer.maths.qmul.ac.uk/Atlas/v3/> (2001).

INSTITUTE OF MATHEMATICS, GDAŃSK UNIVERSITY, WITA STWOSZA 57, 80-952 GDAŃSK, POLAND

E-mail address: `blaszecz@mat.ug.edu.pl`